

Twist deformation of doubly enlarged Newton-Hooke Hopf algebra

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Abstract

We provide fifteen twist-deformed doubly enlarged Newton-Hooke quantum spaces. In τ approaching infinity limit the twisted doubly enlarged Galilei spaces are obtained as well.

1 Introduction

Recently, in article [1] there has been proposed the so-called classical doubly enlarged Newton-Hooke Hopf algebra $\mathcal{U}_0(\widehat{NH}_\pm)$. It contains, apart from rotation (M_{ij}), boost (K_i), acceleration-like (F_i) and space-time translation (P_i, H) generators, the additional ones denoted by R_i . Its algebraic part looks as follows¹

$$\begin{aligned}
[M_{ij}, M_{kl}] &= i(\delta_{il} M_{jk} - \delta_{jl} M_{ik} + \delta_{jk} M_{il} - \delta_{ik} M_{jl}) \quad , \quad [H, P_i] = \pm \frac{i}{\tau^2} K_i \quad , \\
[M_{ij}, K_k] &= i(\delta_{jk} K_i - \delta_{ik} K_j) \quad , \quad [M_{ij}, P_k] = i(\delta_{jk} P_i - \delta_{ik} P_j) \quad , \\
[M_{ij}, H] &= [K_i, K_j] = [K_i, P_j] = 0 \quad , \quad [K_i, H] = -i P_i \quad , \quad [P_i, P_j] = 0 \quad , \quad (1) \\
[F_i, F_j] &= [F_i, P_j] = [F_i, K_j] = 0 \quad , \quad [M_{ij}, F_k] = i(\delta_{jk} F_i - \delta_{ik} F_j) \quad , \\
[R_i, R_j] &= [R_i, P_j] = [R_i, K_j] = [R_i, F_j] = 0 \quad , \quad [M_{ij}, R_k] = i(\delta_{jk} R_i - \delta_{ik} R_j) \quad , \\
[H, F_i] &= 2i K_i \quad , \quad [H, R_i] = 3i F_i \quad ,
\end{aligned}$$

while the coproduct and antipode take the form

$$\Delta_0(a) = a \otimes 1 + 1 \otimes a \quad , \quad S_0(a) = -a \quad . \quad (2)$$

with $a = M_{ij}, K_i, F_i, P_i, H$ and R_i . Besides, the corresponding symmetry transformations of nonrelativistic space-time are given by²

$$\begin{aligned}
x_i \longrightarrow & \alpha_{ij} x_j + a_i C_\pm \left(\frac{t}{\tau} \right) + v_i \tau S_\pm \left(\frac{t}{\tau} \right) \pm 2b_i \tau^2 \left(C_\pm \left(\frac{t}{\tau} \right) - 1 \right) + \\
& \pm 6c_i \tau^3 \left(S_\pm \left(\frac{t}{\tau} \right) - \frac{t}{\tau} \right) \quad , \quad (3)
\end{aligned}$$

$$t \longrightarrow t + a_0 \quad . \quad (4)$$

with $C_+[\frac{t}{\tau}] = \cosh[\frac{t}{\tau}]$, $C_-[\frac{t}{\tau}] = \cos[\frac{t}{\tau}]$, $S_+[\frac{t}{\tau}] = \sinh[\frac{t}{\tau}]$, $S_-[\frac{t}{\tau}] = \sin[\frac{t}{\tau}]$, for finite cosmological constant τ , and

$$x_i \longrightarrow \alpha_{ij} x_j + a_i + v_i t + b_i t^2 + c_i t^3 \quad , \quad (5)$$

$$t \longrightarrow t + a_0 \quad , \quad (6)$$

¹Present in the above commutation relations parameter τ denotes the characteristic for Newton-Hooke algebra cosmological time scale.

²Here we take the following assignments of the parameters:

- a_i - spatial translations (generators P_i)
- v_i - boosts (generators K_i)
- b_i - accelerations (generators F_i)
- α_{ij} - $O(d)$ space rotations (generators M_{ij})
- a_0 - time translation (generator H)
- c_i - new (additional) parameters (generators R_i).

in the case of parameter τ approaching infinity.

It should be noted, that the Hopf structure (1), (2) is the largest known explicitly symmetry (quantum) group at nonrelativistic level. By its different contraction schemes we get respectively:

i) For $R_i \rightarrow 0$ - the acceleration-enlarged Newton-Hooke Hopf algebra $\mathcal{U}_0(\widehat{NH}_\pm)$ proposed in [1], [2],

ii) In the case of R_i and F_i generators approaching zero - the (usual) Newton-Hooke quantum group $\mathcal{U}_0(NH_\pm)$ [3],

iii) For $R_i \rightarrow 0$ and $\tau \rightarrow \infty$ - the acceleration-enlarged Galilei Hopf structure $\mathcal{U}_0(\widehat{G})$ provided in [4],

iv) For both R_i and F_i operators approaching zero as well as for parameter τ running to infinity - the (usual) Galilei quantum group $\mathcal{U}_0(G)$,

and, finally

v) In the case of $\tau \rightarrow \infty$ - the (new) doubly enlarged Galilei Hopf algebra $\mathcal{U}_0(\widehat{\widehat{G}})$.

In this article we consider the Abelian twist-deformations (see [5]-[7]) of doubly enlarged Newton-Hooke Hopf structure $\mathcal{U}_0(\widehat{\widehat{NH}}_\pm)$. In such a way, in accordance with the contraction schemes i)-iv), we rediscovery the twisted space-times for $\mathcal{U}_\kappa(\widehat{NH}_\pm)$, $\mathcal{U}_\kappa(\widehat{G})$, $\mathcal{U}_\kappa(NH_\pm)$ and $\mathcal{U}_\kappa(G)$ quantum groups provided in articles [8], [9] and [10] respectively³. Besides, we find completely new quantum spaces associated with the Abelian twist factors containing additional generators R_i . In the case of finite value of parameter τ they take the form

$$[t, x_i] = 0 \quad , \quad [x_i, x_j] = if_\pm \left(\frac{t}{\tau} \right) \theta_{ij}(x) \quad , \quad (7)$$

with time-dependent functions

$$f_+ \left(\frac{t}{\tau} \right) = f \left(\sinh \left(\frac{t}{\tau} \right), \cosh \left(\frac{t}{\tau} \right) \right) \quad , \quad f_- \left(\frac{t}{\tau} \right) = f \left(\sin \left(\frac{t}{\tau} \right), \cos \left(\frac{t}{\tau} \right) \right) \quad ,$$

$$\theta_{ij}(x) \sim \theta_{ij} = \text{const} \quad \text{or} \quad \theta_{ij}(x) \sim \theta_{ij}^k x_k \quad ,$$

while for cosmological constants τ approaching infinity, they look as follows

$$[x_\mu, x_\nu] = i\alpha_{\mu\nu}^{\rho_1 \dots \rho_n} x_{\rho_1} \dots x_{\rho_n} \quad , \quad (8)$$

³For general classification of all deformations of relativistic and nonrelativistic Hopf algebras see [12], [13].

with $n = 3, 4, 5$ and 6 .

It should be noted, that the motivation for present studies are (at least) twofold. First of all, in accordance with the contraction procedures i)-v), we consider the Abelian twists of the largest known explicitly nonrelativistic Hopf structure. Consequently, the obtained results permit to analyze the classical as well as quantum particle models defined on the corresponding ("largest") twisted space-times (the similar investigations for simpler quantum groups have been performed in articles [14]-[16]).

The paper is organized as follows. In second section fifteen Abelian classical r -matrices for twisted doubly enlarged Newton-Hooke Hopf algebras are considered. The corresponding fifteen quantum space-times are provided in section 3, while their $\tau \rightarrow \infty$ contractions to the doubly enlarged Galilei spaces are discussed in section 4. The final remarks are presented in the last section.

2 Twisted doubly enlarged Newton-Hooke Hopf algebras

In accordance with Drinfeld twist procedure [5]-[7], the algebraic sector of twisted doubly enlarged Newton-Hooke Hopf algebra $\mathcal{U}_0(\widehat{\widehat{NH}}_{\pm})$ remains undeformed (see (1)), while the coproducts and antipodes transform as follows (see formula (2))

$$\Delta_0(a) \rightarrow \Delta.(a) = \mathcal{F}. \circ \Delta_0(a) \circ \mathcal{F}.^{-1} \quad , \quad S.(a) = u. S_0(a) u.^{-1} \quad , \quad (9)$$

with $u. = \sum f_{(1)} S_0(f_{(2)})$ (we use Sweedler's notation $\mathcal{F}. = \sum f_{(1)} \otimes f_{(2)}$). Besides, it should be noted, that the twist factor $\mathcal{F}. \in \mathcal{U}(\widehat{\widehat{NH}}_{\pm}) \otimes \mathcal{U}(\widehat{\widehat{NH}}_{\pm})$ satisfies the classical cocycle condition

$$\mathcal{F}_{.12} \cdot (\Delta_0 \otimes 1) \mathcal{F}. = \mathcal{F}_{.23} \cdot (1 \otimes \Delta_0) \mathcal{F}. \quad , \quad (10)$$

and the normalization condition

$$(\epsilon \otimes 1) \mathcal{F}. = (1 \otimes \epsilon) \mathcal{F}. = 1 \quad , \quad (11)$$

with $\mathcal{F}_{.12} = \mathcal{F}. \otimes 1$ and $\mathcal{F}_{.23} = 1 \otimes \mathcal{F}.$.

It is well known, that the twisted algebra $\mathcal{U}(\widehat{\widehat{NH}}_{\pm})$ can be described in terms of so-called classical r -matrix $r \in \mathcal{U}(\widehat{\widehat{NH}}_{\pm}) \otimes \mathcal{U}(\widehat{\widehat{NH}}_{\pm})$, which satisfies the classical Yang-Baxter equation (CYBE)

$$[[r., r.]] = [r_{.12}, r_{.13} + r_{.23}] + [r_{.13}, r_{.23}] = 0 \quad , \quad (12)$$

where symbol $[[\cdot, \cdot]]$ denotes the Schouten bracket and for $r = \sum_i a_i \otimes b_i$

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1 \quad , \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i \quad , \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i \quad .$$

In this article we consider fifteen Abelian twist-deformations of doubly enlarged Newton-Hooke Hopf algebra, described by the following r -matrices⁴

$$1) \quad r_{\beta_1} = \frac{1}{2}\beta_1^{kl}R_k \wedge R_l \quad [\beta_1^{kl} = -\beta_1^{lk}], \quad (13)$$

$$2) \quad r_{\beta_2} = \frac{1}{2}\beta_2^{kl}R_k \wedge F_l \quad [\beta_2^{kl} = -\beta_2^{lk}], \quad (14)$$

$$3) \quad r_{\beta_3} = \frac{1}{2}\beta_3^{kl}R_k \wedge P_l \quad [\beta_3^{kl} = -\beta_3^{lk}], \quad (15)$$

$$4) \quad r_{\beta_4} = \frac{1}{2}\beta_4^{kl}K_k \wedge R_l \quad [\beta_4^{kl} = -\beta_4^{lk}], \quad (16)$$

$$5) \quad r_{\beta_5} = \beta_5 R_m \wedge M_{kl} \quad [m, k, l - \text{fixed}, \quad m \neq k, l], \quad (17)$$

$$6) \quad r_{\beta_6} = \frac{1}{2}\beta_6^{kl}F_k \wedge F_l \quad [\beta_6^{kl} = -\beta_6^{lk}], \quad (18)$$

$$7) \quad r_{\beta_7} = \frac{1}{2}\beta_7^{kl}F_k \wedge P_l \quad [\beta_7^{kl} = -\beta_7^{lk}], \quad (19)$$

$$8) \quad r_{\beta_8} = \frac{1}{2}\beta_8^{kl}K_k \wedge F_l \quad [\beta_8^{kl} = -\beta_8^{lk}], \quad (20)$$

$$9) \quad r_{\beta_9} = \beta_9 F_m \wedge M_{kl} \quad [m, k, l - \text{fixed}, \quad m \neq k, l], \quad (21)$$

$$10) \quad r_{\beta_{10}} = \frac{1}{2}\beta_{10}^{kl}P_k \wedge P_l \quad [\beta_{10}^{kl} = -\beta_{10}^{lk}], \quad (22)$$

$$11) \quad r_{\beta_{11}} = \frac{1}{2}\beta_{11}^{kl}K_k \wedge P_l \quad [\beta_{11}^{kl} = -\beta_{11}^{lk}], \quad (23)$$

$$12) \quad r_{\beta_{12}} = \frac{1}{2}\beta_{12}^{kl}K_k \wedge K_l \quad [\beta_{12}^{kl} = -\beta_{12}^{lk}], \quad (24)$$

$$13) \quad r_{\beta_{13}} = \beta_{13}K_m \wedge M_{kl} \quad [m, k, l - \text{fixed}, \quad m \neq k, l], \quad (25)$$

$$14) \quad r_{\beta_{14}} = \beta_{14}P_m \wedge M_{kl} \quad [m, k, l - \text{fixed}, \quad m \neq k, l], \quad (26)$$

$$15) \quad r_{\beta_{15}} = \beta_{15}M_{ij} \wedge H. \quad (27)$$

Due to Abelian character of the above carriers (all of them arise from the mutually commuting elements of the algebra), the corresponding twist factors can be get in a standard way [5]-[7], i.e. they take the form

$$\mathcal{F}_{\beta_k} = \exp(ir_{\beta_k}) \quad ; \quad k = 1, 2, \dots, 15. \quad (28)$$

⁴ $a \wedge b = a \otimes b - b \otimes a.$

Let us note that first five matrices include (new) generators R_i , while the next ten factors are the same as in the case of acceleration-enlarged Newton-Hooke Hopf algebra considered in [8]. Of course, for all deformation parameters β_i approaching zero the discussed above Hopf structures $\mathcal{U}_{\beta_i}(\widehat{\widehat{NH}}_{\pm})$ become classical, i.e. they become undeformed.

3 Quantum doubly enlarged Newton-Hooke space-times

Let us now turn to the deformed space-times corresponding to the twist-deformations 1)-15) discussed in pervious section. They are defined as the quantum representation spaces (Hopf modules) for quantum doubly enlarged Newton-Hooke algebras, with action of the deformed symmetry generators satisfying suitably deformed Leibnitz rules [17]-[19].

The action of generators M_{ij} , K_i , P_i , F_i , H and R_i on a Hopf module of functions depending on space-time coordinates (t, x_i) is given by

$$H \triangleright f(t, \bar{x}) = i\partial_t f(t, \bar{x}) \quad , \quad P_i \triangleright f(t, \bar{x}) = iC_{\pm} \left(\frac{t}{\tau} \right) \partial_i f(t, \bar{x}) \quad , \quad (29)$$

$$M_{ij} \triangleright f(t, \bar{x}) = i(x_i \partial_j - x_j \partial_i) f(t, \bar{x}) \quad , \quad K_i \triangleright f(t, \bar{x}) = i\tau S_{\pm} \left(\frac{t}{\tau} \right) \partial_i f(t, \bar{x}) \quad , \quad (30)$$

$$F_i \triangleright f(t, \bar{x}) = \pm 2i\tau^2 \left(C_{\pm} \left(\frac{t}{\tau} \right) - 1 \right) \partial_i f(t, \bar{x}) \quad , \quad (31)$$

and

$$R_i \triangleright f(t, \bar{x}) = \pm 6i\tau^3 \left(S_{\pm} \left(\frac{t}{\tau} \right) - \frac{t}{\tau} \right) \quad . \quad (32)$$

Moreover, the \star -multiplication of arbitrary two functions is defined as follows

$$f(t, \bar{x}) \star_{\beta_i} g(t, \bar{x}) := \omega \circ (\mathcal{F}_{\beta_i}^{-1} \triangleright f(t, \bar{x}) \otimes g(t, \bar{x})) \quad , \quad (33)$$

where symbol \mathcal{F}_{β_i} denotes the twist factor (see (28)) corresponding to the proper doubly enlarged Newton-Hooke Hopf algebra and $\omega \circ (a \otimes b) = a \cdot b$.

In such a way we get fifteen quantum space-times

$$1) \quad [t, x_a]_{\star_{\beta_1}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_1}} = 36i\beta_1^{kl}\tau^6 \left(S_{\pm} \left(\frac{t}{\tau} \right) - \frac{t}{\tau} \right)^2 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (34)$$

$$2) \quad [t, x_a]_{\star_{\beta_2}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_2}} = 12i\beta_2^{kl}\tau^5 \left(S_{\pm} \left(\frac{t}{\tau} \right) - \frac{t}{\tau} \right) \left(C_{\pm} \left(\frac{t}{\tau} \right) - 1 \right) (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (35)$$

$$3) \quad [t, x_a]_{\star_{\beta_3}} = 0 \quad ,$$

$$[x_a, x_b]_{\star_{\beta_3}} = \pm 6i\beta_3^{kl}\tau^3 \left(S_{\pm} \left(\frac{t}{\tau} \right) - \frac{t}{\tau} \right) C_{\pm} \left(\frac{t}{\tau} \right) (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) , \quad (36)$$

$$4) \quad [t, x_a]_{\star_{\beta_4}} = 0 ,$$

$$[x_a, x_b]_{\star_{\beta_4}} = \pm 6i\beta_4^{kl}\tau^4 \left(S_{\pm} \left(\frac{t}{\tau} \right) - \frac{t}{\tau} \right) S_{\pm} \left(\frac{t}{\tau} \right) (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) , \quad (37)$$

$$5) \quad [t, x_a]_{\star_{\beta_5}} = 0 , \quad (38)$$

$$[x_a, x_b]_{\star_{\beta_5}} = \pm 12i\beta_5\tau^3 \left(S_{\pm} \left(\frac{t}{\tau} \right) - \frac{t}{\tau} \right) [\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ik})] (39)$$

$$6) \quad [t, x_a]_{\star_{\beta_6}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_6}} = 4i\beta_6^{kl}\tau^4 \left(C_{\pm} \left(\frac{t}{\tau} \right) - 1 \right)^2 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) , \quad (40)$$

$$7) \quad [t, x_a]_{\star_{\beta_7}} = 0 ,$$

$$[x_a, x_b]_{\star_{\beta_7}} = \pm i\beta_7^{kl}\tau^2 \left(C_{\pm} \left(\frac{t}{\tau} \right) - 1 \right) C_{\pm} \left(\frac{t}{\tau} \right) (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) , \quad (41)$$

$$8) \quad [t, x_a]_{\star_{\beta_8}} = 0 ,$$

$$[x_a, x_b]_{\star_{\beta_8}} = \pm i\beta_8^{kl}\tau^3 \left(C_{\pm} \left(\frac{t}{\tau} \right) - 1 \right) S_{\pm} \left(\frac{t}{\tau} \right) (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) , \quad (42)$$

$$9) \quad [t, x_a]_{\star_{\beta_9}} = 0 ,$$

$$[x_a, x_b]_{\star_{\beta_9}} = \pm 4i\beta_9\tau^2 \left(C_{\pm} \left(\frac{t}{\tau} \right) - 1 \right) [\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ik})] \quad (43)$$

$$10) \quad [t, x_a]_{\star_{\beta_{10}}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_{10}}} = i\beta_{10}^{kl} C_{\pm}^2 \left(\frac{t}{\tau} \right) (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) , \quad (44)$$

$$11) \quad [t, x_a]_{\star_{\beta_{11}}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_{11}}} = i\beta_{11}^{kl}\tau C_{\pm} \left(\frac{t}{\tau} \right) S_{\pm} \left(\frac{t}{\tau} \right) (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) , \quad (45)$$

$$12) \quad [t, x_a]_{\star_{\beta_{12}}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_{12}}} = i\beta_{12}^{kl}\tau^2 S_{\pm}^2 \left(\frac{t}{\tau} \right) (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) , \quad (46)$$

$$13) \quad [t, x_a]_{\star_{\beta_{13}}} = 0 ,$$

$$[x_a, x_b]_{\star_{\beta_{13}}} = 2i\beta_{13}\tau S_{\pm} \left(\frac{t}{\tau} \right) [\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})] , \quad (47)$$

$$14) \quad [t, x_a]_{\star_{\beta_{14}}} = 0 ,$$

$$[x_a, x_b]_{\star_{\beta_{14}}} = 2i\beta_{14} C_{\pm} \left(\frac{t}{\tau} \right) [\delta_{ma}(x_k \delta_{bl} - x_l \delta_{bk}) - \delta_{mb}(x_k \delta_{al} - x_l \delta_{ak})] , \quad (48)$$

$$15) \quad [t, x_a]_{\star_{\beta_{15}}} = 2i\beta_{15} [\delta_{ia}x_j - x_i\delta_{ja}] \quad , \quad [x_a, x_b]_{\star_{\beta_{15}}} = 0 , \quad (49)$$

associated with matrices 1)-15), respectively.

Let us note that due to the form of functions $C_{\pm}[\frac{t}{\tau}]$ and $S_{\pm}[\frac{t}{\tau}]$ the spatial noncommutativities 1)-14) are expanding or periodic in time respectively. Moreover, all of them introduce classical time and quantum spatial directions. The last type of space-time noncommutativity provides the quantum time and classical spatial variables. It should be also noted that in the contraction limit i) and v) we get the twisted acceleration-enlarged Newton-Hooke and usual Newton-Hooke spaces, respectively.

Of course, for all deformation parameters β_i approaching zero, the above quantum space-times become commutative.

4 Twisted doubly enlarged Galilei Hopf algebras - the $\tau \rightarrow \infty$ limit (contraction v))

In this section we provide twisted doubly enlarged Galilei Hopf algebras $\mathcal{U}_{\beta_i}(\widehat{\widehat{G}})$ and corresponding quantum space-times, as the $\tau \rightarrow \infty$ limit of Hopf structures discussed in pervious sections. In such a limit the commutation relations (1) become τ -independent, i.e. we neglect the impact of the cosmological time scale τ on the structure of the considered Hopf algebras.

First of all, we perform the contraction limit $\tau \rightarrow \infty$ of the formulas (1) and (13)-(27). Consequently, the corresponding classical r -matrices remain the same as (13)-(27), while the algebraic sector of all considered $\mathcal{U}_{\beta_i}(\widehat{\widehat{G}})$ algebras takes the form

$$\begin{aligned} [M_{ij}, M_{kl}] &= i(\delta_{il} M_{jk} - \delta_{jl} M_{ik} + \delta_{jk} M_{il} - \delta_{ik} M_{jl}) \quad , \quad [H, P_i] = 0 \quad , \\ [M_{ij}, K_k] &= i(\delta_{jk} K_i - \delta_{ik} K_j) \quad , \quad [M_{ij}, P_k] = i(\delta_{jk} P_i - \delta_{ik} P_j) \quad , \\ [M_{ij}, H] &= [K_i, K_j] = [K_i, P_j] = 0 \quad , \quad [K_i, H] = -iP_i \quad , \quad [P_i, P_j] = 0 \quad , \\ [F_i, F_j] &= [F_i, P_j] = [F_i, K_j] = 0 \quad , \quad [M_{ij}, F_k] = i(\delta_{jk} F_i - \delta_{ik} F_j) \quad , \\ [R_i, R_j] &= [R_i, P_j] = [R_i, K_j] = [R_i, F_j] = 0 \quad , \quad [M_{ij}, R_k] = i(\delta_{jk} R_i - \delta_{ik} R_j) \quad , \\ [H, F_i] &= 2iK_i \quad , \quad [H, R_i] = 3iF_i \quad . \end{aligned} \quad (50)$$

The corresponding coproduct sectors can be get by application of the formulas (9) and (28).

Let us now turn to the corresponding quantum nonrelativistic space-times. One can check (see $\tau \rightarrow \infty$ limit of the formulas (34)-(49)) that they look as follows⁵

$$1) \quad [t, x_a]_{\star_{\beta_1}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_1}} = i\beta_1^{kl} t^6 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (51)$$

$$2) \quad [t, x_a]_{\star_{\beta_2}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_2}} = i\beta_2^{kl} t^5 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (52)$$

$$3) \quad [t, x_a]_{\star_{\beta_3}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_3}} = i\beta_3^{kl} t^3 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (53)$$

$$4) \quad [t, x_a]_{\star_{\beta_4}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_4}} = i\beta_4 t^4 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (54)$$

$$5) \quad [t, x_a]_{\star_{\beta_5}} = 0 \quad ,$$

$$[x_a, x_b]_{\star_{\beta_5}} = 2i\beta_5^{kl} t^3 [\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})] \quad , \quad (55)$$

$$6) \quad [t, x_a]_{\star_{\beta_6}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_6}} = i\beta_6^{kl} t^4 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (56)$$

$$7) \quad [t, x_a]_{\star_{\beta_7}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_7}} = \frac{i}{2}\beta_7^{kl} t^2 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (57)$$

$$8) \quad [t, x_a]_{\star_{\beta_8}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_8}} = \frac{i}{2}\beta_8^{kl} t^3 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (58)$$

$$9) \quad [t, x_a]_{\star_{\beta_9}} = 0 \quad ,$$

$$[x_a, x_b]_{\star_{\beta_9}} = 2i\beta_9 t^2 [\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})] \quad , \quad (59)$$

$$10) \quad [t, x_a]_{\star_{\beta_{10}}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_{10}}} = i\beta_{10}^{kl} (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (60)$$

$$11) \quad [t, x_a]_{\star_{\beta_{11}}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_{11}}} = i\beta_{11}^{kl} t (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (61)$$

$$12) \quad [t, x_a]_{\star_{\beta_{12}}} = 0 \quad , \quad [x_a, x_b]_{\star_{\beta_{12}}} = i\beta_{12}^{kl} t^2 (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) \quad , \quad (62)$$

$$13) \quad [t, x_a]_{\star_{\beta_{13}}} = 0 \quad ,$$

$$[x_a, x_b]_{\star_{\beta_{13}}} = 2i\beta_{13} t [\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})] \quad , \quad (63)$$

$$14) \quad [t, x_a]_{\star_{\beta_{14}}} = 0 \quad ,$$

⁵It should be noted that the commutation relations (51)-(65) can be also derived with use of the formula (33) and differential representation of doubly enlarged Galilei generators.

$$[x_a, x_b]_{\star_{\beta_{14}}} = 2i\beta_{14} [\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})] , \quad (64)$$

$$15) \quad [t, x_a]_{\star_{\beta_{15}}} = 2i\beta_{15} [\delta_{ia}x_j - x_i\delta_{ja}] \quad , \quad [x_a, x_b]_{\star_{\beta_{15}}} = 0 , \quad (65)$$

in the case of $\mathcal{U}_{\beta_1}(\widehat{G}), \dots, \mathcal{U}_{\beta_{15}}(\widehat{G})$ Hopf algebras respectively.

Obviously, for all deformation parameters β_i approaching zero the above Hopf algebras become classical, while the corresponding quantum space-times - commutative.

5 Final remarks

In this article we consider fifteen Abelian twist-deformations of doubly enlarged Newton-Hooke Hopf algebras. Besides, we demonstrate that as in the case of twist-deformed acceleration-enlarged Newton-Hooke Hopf algebra, the corresponding spaces can be periodic and expanding in time for $\mathcal{U}_{\beta_i}(\widehat{NH}_-)$ and $\mathcal{U}_{\beta_i}(\widehat{NH}_+)$ quantum groups respectively. In $\tau \rightarrow \infty$ limit we also discover new twisted doubly enlarged Galilei space-times (51)-(65).

It should be noted that present studies can be extended in various ways. First of all, one can find the dual Hopf structures $\mathcal{D}_{\beta_i}(\widehat{NH}_{\pm})$ with the use of FRT procedure [20] or by canonical quantization of the corresponding Poisson-Lie structures [21]. Besides, as it was already mentioned in Introduction, one should ask about the basic dynamical models corresponding to the doubly enlarged Newton-Hooke space-times (34)-(49). Finally, one can also consider more complicated (non-Abelian) twist deformations of doubly enlarged Newton-Hooke Hopf algebras, i.e. one can find the twisted coproducts, corresponding noncommutative space-times and dual Hopf structures. Such problems are now under consideration.

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References

- [1] J. Gomis, J. Lukierski, Phys. Lett. B 664, 107 (2008); arXiv: 0803.3965 [hep-th]
- [2] J. Lukierski, P.C. Stichel, W.J. Zakrzewski, "Acceleration-enlarged symmetries in nonrelativistic space-time with a cosmological constant"; arXiv: 0710.3093 [hep-th]
- [3] H. Bacry, J.M. Levy-Leblond, J. Math. Phys. 9, 1605 (1968)
- [4] J. Lukierski, P.C. Stichel, W.J. Zakrzewski, Phys. Lett. B 650, 203 (2007)

- [5] V.G. Drinfeld, Soviet Math. Dokl. 32, 254 (1985); Algebra i Analiz (in Russian) 1, Fasc. 6, p. 114 (1989)
- [6] V. Chari, A. Pressley, "*A Guide to Quantum Groups*", Cambridge University Press, Cambridge, 1994
- [7] S. Majid, "*Foundations of quantum group theory*", Cambridge University Press, 2000
- [8] M. Daszkiewicz, Acta Phys. Polon. B 41, 1889 (2010); arXiv:1007.4654 [math-ph]
- [9] M. Daszkiewicz, Mod. Phys. Lett. A 24, 1325 (2009); arXiv: 0904.0432 [hep-th]
- [10] M. Daszkiewicz, Mod. Phys. Lett. A 23, 505 (2008), IFT-UWR-LV-420; arXiv: 0801.1206 [hep-th]
- [11] M. Daszkiewicz, Mod. Phys. Lett. A 23, 1757 (2008); arXiv:0807.0133 [hep-th]
- [12] S. Zakrzewski, *Poisson Structures on the Poincare group*; q-alg/9602001
- [13] Y. Brihaye, E. Kowalczyk, P. Maslanka, Poisson-Lie structure on Galilei group; math/0006167
- [14] J.M. Romero, J.A. Santiago, J.D. Vergara, Phys. Lett. A 310, 9 (2003); arXiv:hep-th/0211165
- [15] M. Daszkiewicz and C.J. Walczyk, Phys. Rev. D 77, 105008 (2008); arXiv:0802.3575 [math-ph]
- [16] Y.G. Miao, X.D. Wang, S.J. Yu, Annals Phys. 326, 2091 (2011); arXiv:0911.5227 [math-ph]
- [17] C. Blohmann, J. Math. Phys. 44, 4736 (2003); q-alg/0209180
- [18] J. Wess, "*Deformed coordinate spaces: Derivatives*"; hep-th/0408080
- [19] M. Chaichian, P.P. Kulish, K. Nashijima, A. Tureanu, Phys. Lett. B 604, 98 (2004)
- [20] L.D. Faddeev, N.Yu. Reshetikhin, L.A. Takhtadzhyan, Leningrad Math. J. 1, 178 (1990)
- [21] L.A. Takhtajan, "*Introduction to Quantum Groups*"; in Clausthal Proceedings, Quantum groups 3-28 (see High Energy Physics Index 29 (1991) No. 12256)